

New results on lower bounds for the number of $(\leq k)$ -facets ^{*}

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Abstract

In this paper we present three different results dealing with the number of $(\leq k)$ -facets of a set of points:

1. We give structural properties of sets in the plane that achieve the optimal lower bound $3\binom{k+2}{2}$ of $(\leq k)$ -edges for a fixed $0 \leq k \leq \lfloor n/3 \rfloor - 1$;
2. We give a simple construction showing that the lower bound $3\binom{k+2}{2} + 3\binom{k - \lfloor \frac{n}{3} \rfloor + 2}{2}$ for the number of $(\leq k)$ -edges of a planar point set appeared in [Aichholzer et al. New lower bounds for the number of $(\leq k)$ -edges and the rectilinear crossing number of K_n . *Disc. Comput. Geom.* 38:1 (2007), 1–14] is optimal in the range $\lfloor n/3 \rfloor \leq k \leq \lfloor 5n/12 \rfloor - 1$;
3. We show that for $k < \lfloor n/(d+1) \rfloor$ the number of $(\leq k)$ -facets of a set of n points in general position in \mathbb{R}^d is at least $(d+1)\binom{k+d}{d}$, and that this bound is tight in the given range of k .

1 Introduction

In this paper we deal with the problem of giving lower bounds to the number of $(\leq k)$ -facets of a set of points S : An oriented simplex with vertices at points of S is said to be a k -facet of S if it has exactly k points in the positive side of its affine hull. Similarly, the

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simplex is said to be an $(\leq k)$ -*facet* if it has at most k points in the positive side of its affine hull. If $S \subset \mathbb{R}^2$, a k -facet of S is usually named a k -edge.

The number of k -facets of S is denoted by $e_k(S)$, and $E_k(S) = \sum_{j=0}^k e_j(S)$ is the number of $(\leq k)$ -facets (the set S will be omitted when it is clear from the context). Giving bounds on these quantities, and on the number of the companion concept of k -sets, is one of the central problems in Discrete and Computational Geometry, and has a long history that we will not try to summarize here. Chapter 8.3 in [5] is a complete and up to date survey of results and open problems in the area.

Regarding lower bounds for $E_k(S)$, which is the main topic of this paper, the problem was first studied by Edelsbrunner et al. [7] due to its connections with the complexity of higher order Voronoi diagrams. In that paper it was stated that, in \mathbb{R}^2 ,

$$E_k(S) \geq 3 \binom{k+2}{2} \quad (1)$$

and it was given an example showing tightness for $0 \leq k \leq \lfloor n/3 \rfloor - 1$. The proof used circular sequences but, unfortunately, contained an unpluggable gap, as pointed out by Lovász et al. [9]. A correct proof, also using circular sequences, was independently found by Ábrego and Fernández-Merchant [1] and Lovász et al. [9]. In both papers a strong connection was discovered between the number of $(\leq k)$ -edges and the number of convex quadrilaterals in a point set S . Specifically, if $\square(S)$ denotes the number of convex quadrilaterals in S , in [9] it was shown that

$$\square(S) = \sum_{k < \frac{n-2}{2}} (n - 2k - 3) E_k(S) - \frac{3}{4} \binom{n}{3} + c_n, \quad (2)$$

where

$$c_n = \begin{cases} \frac{1}{4} E_{\frac{n-3}{2}}(S), & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Giving lower bounds for $\square(S)$ is in turn equivalent to determining the rectilinear crossing number of the complete graph: if we draw K_n on top of a set of points S , then the number of intersections in the drawing is exactly the number of convex quadrilaterals in S . The interested reader can go through the extensive online bibliography by Vrt'o [10] where the focus is on the problem of crossing numbers of graphs.

The lower bound in Equation 1 was slightly improved for $k \geq \lfloor \frac{n}{3} \rfloor$ by Balogh and Salazar [4], again using circular sequences. Using different techniques, and based on the observation that it suffices to proof the bound for sets with triangular convex hull, we have recently shown [2] that, in \mathbb{R}^2 ,

$$E_k(S) \geq 3 \binom{k+2}{2} + \sum_{j=\lfloor \frac{n}{3} \rfloor}^k (3j - n + 3). \quad (3)$$

If n is divisible by 3, this expression can be written as

$$E_k(S) \geq 3 \binom{k+2}{2} + \binom{k - \frac{n}{3} + 2}{2}.$$

In this paper we deal with three different problems related to lower bounds for E_k : In Section 2, we study structural properties of those sets in \mathbb{R}^2 that achieve the lower

bound in Equation 1 for a fixed $0 \leq k \leq \lfloor n/3 \rfloor - 1$. The main result of this section is that if $E_k(S)$ is minimum for a given k , then $E_j(S)$ is also minimum for every $0 \leq j < k$. In Section 3 we give a construction which shows tightness of the lower bound in Equation 3 in the range $\lfloor n/3 \rfloor \leq k \leq \lfloor 5n/12 \rfloor - 1$. Finally, in Section 4 we study the d -dimensional version of the problem and show that, for a set of n points in general position in \mathbb{R}^d ,

$$E_k(S) \geq (d+1) \binom{k+d}{d}, \text{ for } 0 \leq k < \lfloor \frac{n}{d+1} \rfloor, \quad (4)$$

and that this bound is tight in that range. To the best of our knowledge, this is the first result of this kind in \mathbb{R}^d .

2 Optimal sets for $(\leq k)$ -edge vectors

Given $S \subset \mathbb{R}^2$, let us denote by $\mathcal{E}_k(S)$ the set of all $(\leq k)$ -edges of S , hence $E_k(S)$ is the cardinality of $\mathcal{E}_k(S)$. Throughout this section we consider $k \leq \lfloor \frac{n}{3} \rfloor - 1$. Recall that for a fixed such k , $E_k(S)$ is optimal if $E_k(S) = 3 \binom{k+2}{2}$. Recall also that, by definition, a j -edge has exactly j points of S in the positive side of its affine hull, which in this case is the open half plane to the right of its supporting line.

We start by giving a new, simple, and self-contained proof of the bound in Equation 1, using a new technique which will be useful in the rest of the section. Although in this section they will be used in \mathbb{R}^2 , the following notions are presented in \mathbb{R}^d for the sake of generality and in view of Section 4.

Definition 1 ([8]). Let S be a set of n points and \mathcal{H} a family of sets in \mathbb{R}^d . A subset $N \subset S$ is called an ϵ -net of S (with respect to \mathcal{H}) if for every $H \in \mathcal{H}$ such that $|H \cap S| > \epsilon n$ we have that $H \cap N \neq \emptyset$.

Definition 2. A *simplicial ϵ -net* of $S \subset \mathbb{R}^d$ is a set of $d+1$ vertices of the convex hull of S that are an ϵ -net of S with respect to closed half-spaces. A simplicial $\frac{1}{2}$ -net will be called a *simplicial half-net*.

Lemma 3. *Every set $S \subset \mathbb{R}^2$ of n points has a simplicial half-net.*

Proof. Let T be a triangle spanned by three vertices of the convex hull of S . An edge e of T is called *good* if the closed half plane of its supporting line which contains the third vertex of T , contains at least $\frac{n}{2}$ points from S . T is called *good* if it consists of three good edges. Clearly, the vertices of a good triangle are a simplicial half-net of S .

Let T be an arbitrary triangle spanned by vertices of the convex hull of S and assume that T is not good. Then observe that only one edge e of T is not good and let v be the vertex of T not incident to e . Choose a point v' of the convex hull of S opposite to v with respect to e . Then e and v' induce a triangle T' in which e is a good edge. If T' is a good triangle we are done. Otherwise we iterate this process. As the cardinalities of the subsets of vertices of S considered are strictly decreasing (the subsets being restricted by the half plane induced by e), the process terminates with a good triangle. \square

Theorem 4. *For every set S of n points and $0 \leq k < \lfloor \frac{n-2}{2} \rfloor$ we have $E_k(S) \geq 3 \binom{k+2}{2}$.*

Proof. The proof goes by induction on n . From Lemma 3, we can guarantee the existence of $T = \{a, b, c\} \subset S$, an $\frac{1}{2}$ -net made up with vertices of the convex hull.

Let $S' = S \setminus T$ and consider an edge $e \in \mathcal{E}_{k-2}(S')$. We observe that T cannot be to the right of e : there are at least $\frac{n}{2}$ points on the closed half-plane to the left of e and that would contradict the definition of $\frac{1}{2}$ -net. Therefore, $e \in \mathcal{E}_k(S)$.

If we denote by $\mathcal{ET}_k(S)$ the set of $(\leq k)$ -edges of S adjacent to points in T , we have that

$$\mathcal{E}_{k-2}(S') \cup \mathcal{ET}_k(S) \subset \mathcal{E}_k(S). \quad (5)$$

There are $2(k+1)$ $(\leq k)$ -edges incident to each of the convex hull vertices a, b, c (which can be obtained rotating a ray based on that vertex). We observe that at most three edges of $\mathcal{ET}_k(S)$ might be incident to two points of T (those of the triangle T) and that the union in Equation 5 is disjoint. Therefore, using the induction hypothesis we have

$$E_k(S) \geq E_{k-2}(S') + 3 + 6k \geq 3 \binom{k}{2} + 3 + 6k = 3 \binom{k+2}{2}. \quad (6)$$

□

Corollary 5. *Let S be a set of n points, $T = \{a, b, c\}$ a simplicial half-net of S and $S' = S \setminus T$. If $E_k(S) = 3 \binom{k+2}{2}$ then:*

(a) $E_{k-2}(S') = 3 \binom{k}{2}$.

(b) A k -edge of S is either a $(k-2)$ -edge of S' or is adjacent to a point in T .

Proof. If $E_k(S) = 3 \binom{k+2}{2}$, both inequalities in Equation 6 are tight. Therefore $E_{k-2}(S') = 3 \binom{k}{2}$ and Equation 5 becomes $\mathcal{E}_{k-2}(S') \cup \mathcal{ET}_k(S) = \mathcal{E}_k(S)$ (disjoint union) which trivially implies part (b). □

Theorem 6. *If $E_k(S) = 3 \binom{k+2}{2}$, then S has a triangular convex hull.*

Proof. We prove the statement by induction over k . For $k = 0$ nothing has to be proven, so let $k = 1$, assume that $E_1 = 9$ and let $h = |CH(S)|$. We have h 0-edges and at least h 1-edges (two per convex hull vertex, but each edge might be counted twice). Thus $E_1 = 9 \geq 2h$ and therefore $h \leq 4$. Assume now $h = 4$. Then at most two 1-edges can be counted twice, namely the two diagonals of the convex hull. Thus we have $4 + 8 - 2 = 10$ (≤ 1) -edges and we conclude that if $E_1 = 9$, then S has a triangular convex hull.

For the general case consider $k \geq 2$, let $T = \{a, b, c\}$ be the simplicial half-net guaranteed by Lemma 3 and let $S' = S \setminus T$. From Corollary 5, part (a), we know that $E_{k-2}(S') = 3 \binom{k}{2}$ and, by induction, we may assume that S' has a triangular convex hull. Moreover, from part (b), no $(k-1)$ -edge of S' can be an $(\leq k)$ -edge of S and, therefore, any $(k-1)$ -edge of S' must have two vertices of T on its positive side. Consider the six $(k-1)$ -edges of S' incident to the three convex hull vertices of S' : See Figure 1, where the supporting lines of these $(k-1)$ -edges are drawn as dashed lines and S' is depicted as the central triangle. Each cell outside S' in the arrangement of the supporting lines contains a number counting the $(k-1)$ -edges considered which have that cell on their positive side. A simple counting argument shows that the only way of placing the three vertices a, b, c of T such that each $(k-1)$ -edge of S' drawn has three of them on its positive side is to place one in each cell labeled with a 4. We conclude that no vertex of S' can be on the convex hull of S and the theorem follows. □

Corollary 7. *If $E_k(S) = 3 \binom{k+2}{2}$, then the outermost $\lceil \frac{k}{2} \rceil$ layers of S are triangles.*

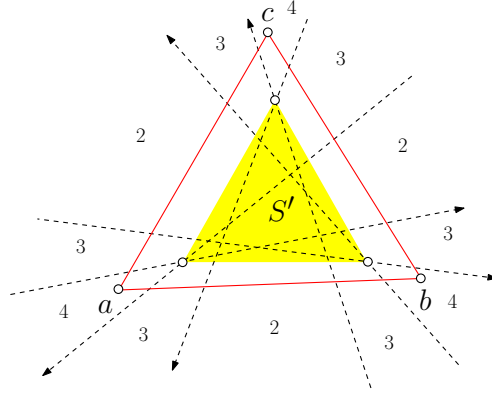


Figure 1: Each $(k-1)$ -edge of S' incident to a convex hull vertex of S' (supporting lines are shown as dashed lines) has two vertices of T on its positive side.

Proof. From the optimality of $E_k(S)$ and using the same argument as in the proof of Theorem 6, it follows that we can iteratively remove the outermost $\lceil \frac{k}{2} \rceil$ layers to obtain optimal subsets, which, by Theorem 6, have triangular convex hulls. \square

Theorem 8. *If $E_k(S) = 3\binom{k+2}{2}$, then $E_j(S) = 3\binom{j+2}{2}$ for every $0 \leq j \leq k$.*

Proof. We prove the theorem by induction on k . For $k = 0, 1$ the theorem is equivalent to Theorem 6, so let $k \geq 2$. It is sufficient to show that optimality of $E_k(S)$ implies optimality of $E_{k-1}(S)$, as the theorem follows by induction.

Let T be the vertices of $CH(S)$ (which is a triangle as guaranteed by Theorem 6) and let $S' = S \setminus T$. As in Theorem 4 we have

$$\mathcal{E}_{k-3}(S') \cup \mathcal{ET}_{k-1}(S) \subset \mathcal{E}_{k-1}(S).$$

Observe that $E_{k-2}(S')$ is optimal, as guaranteed by Corollary 5 and this implies optimality of $E_{k-3}(S')$ by induction. $|\mathcal{ET}_{k-1}(S)|$ is also optimal because the convex hull of S is the triangle T . Therefore, to prove optimality of $E_{k-1}(S)$ it only remains to show that no $(k-2)$ -edge of S' can be a $(k-1)$ -edge of S .

So let e be a $(k-2)$ -edge of S' and let p and q be the vertices of the convex hull of S' incident to e or on its positive side. The existence of p and q is guaranteed by Corollary 5, part (b). Without loss of generality, assume that the edge pq is horizontal with the remaining vertices of S' above it, see Figure 2 for the rest of the proof. Let ℓ_1 be the $(k-1)$ -edge of S' incident to p which has q on its positive side and ℓ_2 the $(k-1)$ -edge incident to q and having p on its positive side. The *boundary chain* is the lower envelope of ℓ_1 , pq , and ℓ_2 . We claim that e does not intersect the boundary chain and lies above it. If e is incident to p or q then the claim is obviously true. Otherwise observe that e has to intersect the supporting lines of both considered $(k-1)$ -edges in the interior of S' , as otherwise there would be too many vertices on the positive side of e . But then again e lies above the boundary chain and the claim follows.

From the proof of Theorem 6 we know that two of the vertices of the convex hull of S have to lie below our boundary chain (below the $(k-1)$ -edges, see a and b in Figure 2) and thus on the positive side of e . Therefore e has at least k vertices of S on its positive side and does not belong to $\mathcal{E}_{k-1}(S)$. We conclude that $E_{k-1}(S)$ is optimal and the theorem follows. \square

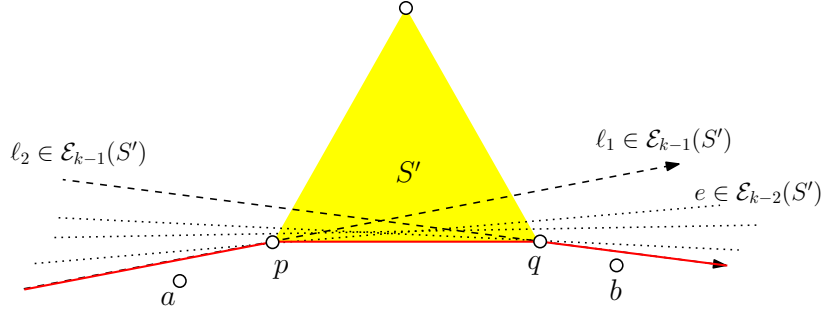


Figure 2: All $(k-2)$ -edges of S' (supporting lines are shown as dotted lines) lie above the (bold) lower envelope.

Corollary 9. *Let $0 \leq k \leq \lfloor \frac{n}{3} \rfloor - 1$. If $E_k(S) = 3 \binom{k+2}{2}$, then $e_j(S) = 3(j+1)$ for $0 \leq j \leq k$.*

3 Tightness of the lower bound for $(\leq k)$ -edges in \mathbb{R}^2

In this section we show a point configuration which proves tightness in the range $0 \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$ of the lower bound for $E_k(S)$ given in [2]. Consider the configuration in Figure 3 (left), which is composed of three rotationally symmetric chains, each one associated to a convex hull vertex, fulfilling the following properties (where left and right are considered with respect to the corresponding convex hull vertex):

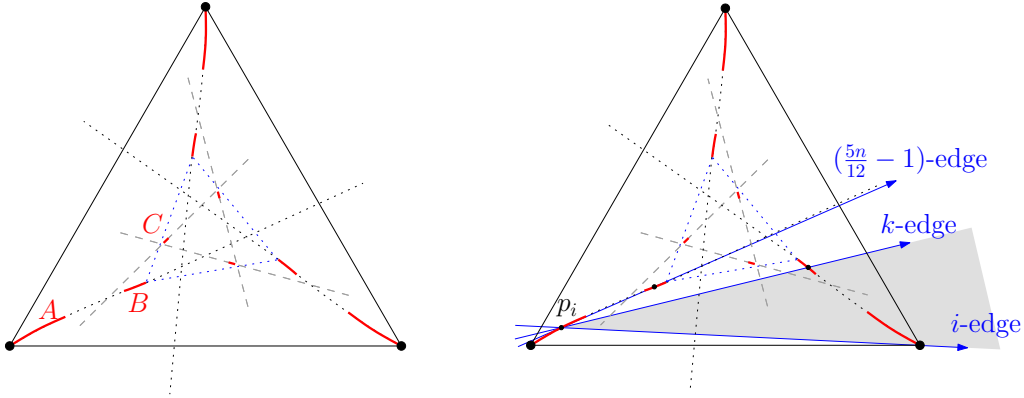


Figure 3: Left: Configuration showing tightness for $0 \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$. Right: For $i \in \{0, \dots, \frac{2n}{12} - 1\}$, exactly one j -edge appears for each $j \in \{i, \dots, k\}$.

- The first part of the chain is slightly convex to the right and contains $\frac{3n}{12}$ points, with a hole between the first $\frac{2n}{12}$ points (which we call subchain A) and the remaining $\frac{n}{12}$ points (called subchain B).
- Each chain is completed with a subchain C , composed of another $\frac{n}{12}$ points slightly convex to the left.
- All the lines spanned by two points in $A \cup B$, oriented from the outermost to the innermost endpoint, leave to the right the next chain in counterclockwise order, and to the left both the points in C and those in the remaining chain.

- All the lines spanned by two points in C separate subchains A and B . Furthermore, when oriented from the outermost to the innermost endpoint, they leave to the right the two remaining subchains of type C .
- The triangle defined by the innermost points of chains of type B contains all the chains of type C .

Theorem 10. For the point configuration S defined above and $\lfloor \frac{n}{3} \rfloor \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$,

$$E_k(S) = 3 \binom{k+2}{2} + 3 \binom{k - \frac{n}{3} + 2}{2},$$

which matches the lower bound stated in [2] for n divisible by 3.

Proof. Because of the rotational symmetry, we can focus on one of the three chains $A \cup B \cup C$ and let p_i be the $(i+1)$ -th point on that chain. We will count oriented j -edges of type $\overrightarrow{p_i q}$ (i.e. with p_i on the tail) for $j \leq k$. In order to do so we rotate counterclockwise a ray based on p_i , starting from the one passing through the convex hull vertex of the next chain in counterclockwise order. Three cases arise, depending on the index i of p_i , which correspond to p_i lying on one of the three subchains:

- (A) For $i \in \{0, \dots, \frac{2n}{12} - 1\}$, exactly one j -edge appears for each j in the range $j \in \{i, \dots, k\}$, while all the remaining j -edges $\overrightarrow{p_i q}$ in the rotation have $j > k$ since at some point we find a $(\frac{5n}{12} - 1)$ -edge and after that all the j -edges found have $j \geq \frac{5n}{12} > k$. See Figure 3 (right).
- (B) For $i \in \{\frac{2n}{12}, \dots, \frac{3n}{12} - 1\}$, exactly one j -edge appears for each j in the ranges $j \in \{i, \dots, k\}$ and $j \in \{\frac{7n}{12} - i - 1, \dots, k\}$, while all the remaining j -edges $\overrightarrow{p_i q}$ in the rotation have $j > k$. See Figure 4.
- (C) For $i \in \{\frac{3n}{12}, \dots, \frac{4n}{12} - 1\}$, exactly one j -edge appears for each j in the ranges $j \in \{i, \dots, k\}$ and $j \in \{\frac{8n}{12} - i - 1, \dots, k\}$, while all the remaining j -edges $\overrightarrow{p_i q}$ in the rotation have $j > k$. See Figure 5.

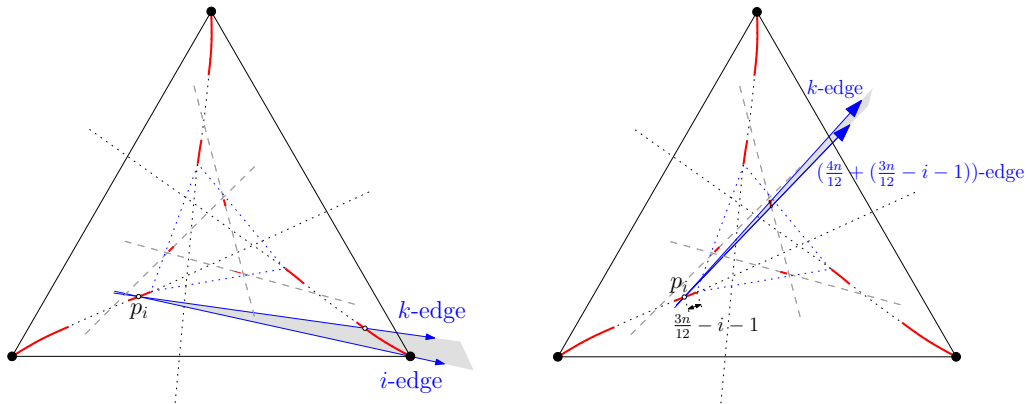


Figure 4: Left: For $i \in \{\frac{2n}{12}, \dots, \frac{3n}{12} - 1\}$, exactly one j -edge appears for each $j \in \{i, \dots, k\}$. Right: For $i \in \{\frac{2n}{12}, \dots, \frac{3n}{12} - 1\}$, exactly one j -edge appears for each $j \in \{\frac{7n}{12} - i - 1, \dots, k\}$.

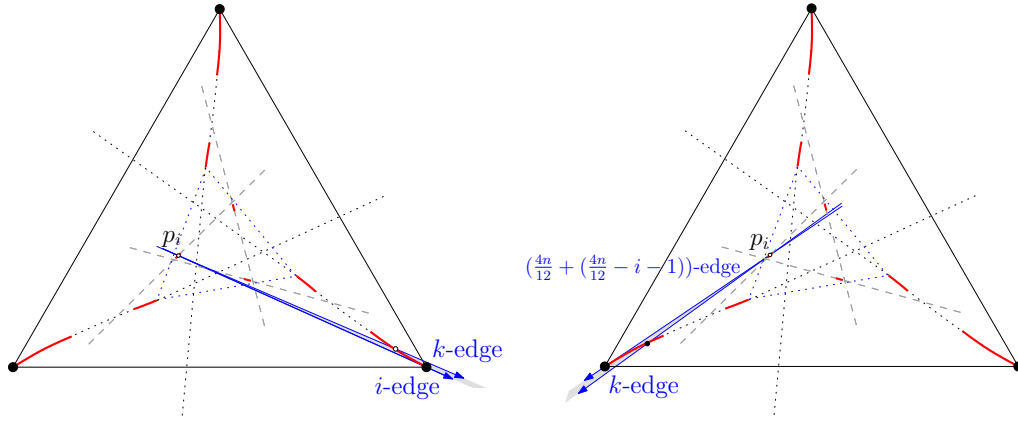


Figure 5: Left: For $i \in \{\frac{3n}{12}, \dots, \frac{4n}{12} - 1\}$, exactly one j -edge appears for each $j \in \{i, \dots, k\}$. Right: For $i \in \{\frac{3n}{12}, \dots, \frac{4n}{12} - 1\}$, exactly one j -edge appears for each $j \in \{\frac{8n}{12} - i - 1, \dots, k\}$.

Let us point out that, depending on the values of k and i , some of the above ranges could actually be empty. Now we are ready to count the total number of $(\leq k)$ -edges incident to points p_i on the first chain, which is:

$$\sum_{i=0}^{\frac{2n}{12}-1} (k-i+1) + \sum_{i=\frac{2n}{12}}^{\frac{3n}{12}-1} (k-i+1) + \sum_{i=\frac{3n}{12}}^{\frac{4n}{12}-1} (k-i+1) + \sum_{i=\frac{2n}{12}}^{\frac{3n}{12}-1} (k - \frac{7n}{12} + i + 2) + \sum_{i=\frac{3n}{12}}^{\frac{4n}{12}-1} (k - \frac{8n}{12} + i + 2),$$

where the first three summands come from the first ranges of the three cases above, while the two remaining summands come from the second ranges in cases (B) and (C). Merging the first three summands and rewriting the two latter ones, the above sum equals

$$\sum_{i=0}^{\frac{4n}{12}-1} (k-i+1) + \sum_{i=\frac{5n}{12}-1}^{\frac{4n}{12}} (k-i+1) + \sum_{i=\frac{5n}{12}-1}^{\frac{4n}{12}} (k-i+1) = \sum_{j=1}^{k+1} j + \sum_{j=1}^{k-\frac{4n}{12}+1} j = \binom{k+2}{2} + \binom{k-\frac{n}{3}+2}{2},$$

where the first equality comes from neglecting the negative summands, due to the above mentioned empty ranges, and merging the first two sums. This result has to be multiplied by the three chains of the configuration, so we get

$$E_k(S) = 3 \binom{k+2}{2} + 3 \binom{k-\frac{n}{3}+2}{2}.$$

□

4 A lower bound for $(\leq k)$ -facets in \mathbb{R}^d

Throughout this section, $S \subset \mathbb{R}^d$ will be a set of n points in general position.

We remind that $e_k(S)$ and $E_k(S)$ denote, respectively, the number of k -facets and the number of $(\leq k)$ -facets of S . The main result of this section is a lower bound for the number of $(\leq k)$ -facets of a set of n points in general position in \mathbb{R}^d in the range $0 \leq k < \lfloor \frac{n}{d+1} \rfloor$.

The proof follows the approach in Theorem 4, using the fact that every set of points has a centerpoint: a point $c \in \mathbb{R}^d$ is a *centerpoint* of S if no open halfspace that avoids c contains more than $\lceil \frac{dn}{d+1} \rceil$ points of S (see [6]).

Theorem 11. *Let S be a set of $n \geq d+1$ points in \mathbb{R}^d in general position. Then*

$$E_k(S) \geq (d+1) \binom{k+d}{d} \quad \text{if } 0 \leq k < \lfloor \frac{n}{d+1} \rfloor.$$

Furthermore, the bound on $E_k(S)$ is tight in the given range of k .

Proof. The proof uses induction on n and d . The base case for $n = d+1$ is obvious and for $d = 2$ is just Equation 1.

Let $k < \lfloor \frac{n}{d+1} \rfloor$ and let c be a centerpoint of S . Let us consider a simplex T with vertices in the convex hull of S and containing c and let $S' = S \setminus T$. From the definition of centerpoint, it follows that no open halfspace that avoids T contains more than $\lceil \frac{dn}{d+1} \rceil - 1$ points or, equivalently, every closed halfspace containing T has at least $\lfloor \frac{n}{d+1} \rfloor + 1$ points.

We denote by $\mathcal{E}_k^j(S)$ the set of $(\leq k)$ -facets of S adjacent to exactly j vertices of T , and $E_k^j(S)$ will be the cardinality of $\mathcal{E}_k^j(S)$.

For $j = 0$, we observe that $\mathcal{E}_{k-d}^0(S') \subset \mathcal{E}_k^0(S)$, because a closed halfspace containing at most k points cannot contain all the vertices of T . Because $k-d \leq \lfloor \frac{n-(d+1)}{d+1} \rfloor - 1$, we can apply induction on n and get

$$E_k^0(S) \geq E_{k-d}^0(S') \geq (d+1) \binom{(k-d)+d}{d} = (d+1) \binom{k}{d}.$$

For $1 \leq j \leq d$, let T_j be a subset of j vertices of T and let S_π be the projection from T_j of $S \setminus T$ onto the $(d-j)$ -dimensional subspace π defined by the points in $T \setminus T_j$: a point $p \in S \setminus T$ is mapped to the intersection between the j -flat defined by p and T_j and the $(d-j)$ -flat defined by points in $T \setminus T_j$. Using the general position assumption, it is easy to see that the intersection has dimension zero. If the intersection were empty, we could slightly perturb p without changing the number of $(\leq k)$ -facets of S .

Now, if $\sigma \subset S_\pi$ is an $(\leq (k-d+j))$ -facet of S_π , then $\sigma \cup T_j$ is an $(\leq k)$ -facet of S (as before, a halfspace containing at most k points of S cannot contain all the vertices of T). Because

$$k-d+j \leq \left\lfloor \frac{n}{d+1} \right\rfloor - 1 \leq \left\lfloor \frac{n-j}{d-j+1} \right\rfloor - 1$$

we can apply induction in d and n , obtaining that there are at least

$$(d-j+1) \binom{k-d+j+(d-j)}{d-j} = (d-j+1) \binom{k}{d-j}$$

$(\leq k)$ -facets of S adjacent to T_j . Summing on all the subsets of j points of T , we get

$$E_k^j(S) \geq \binom{d+1}{j} (d-j+1) \binom{k}{d-j},$$

and, finally,

$$E_k(S) \geq \sum_{j=0}^d \binom{d+1}{j} (d-j+1) \binom{k}{d-j} = (d+1) \binom{k+d}{d}.$$

As for tightness, the example showing that the bound $3\binom{k+2}{2}$ is tight for $0 \leq k \leq \lfloor \frac{n}{3} \rfloor - 1$ in the planar case [7] can be extended to \mathbb{R}^d : Consider $d+1$ rays in \mathbb{R}^d emanating from the origin and with the property that any hyperplane containing one of them leaves on each open halfspace at least one of the remaining rays. For instance, we could take the rays defined by the origin and the vertices of a regular simplex inscribed in the unit d -sphere.

Let $n = (d+1)m$ and put chains C_1, \dots, C_{d+1} with m points on each ray, slightly perturbed to achieve general position. For $j < m$, every j -facet of S is defined by d points on different chains, because a facet defined by two points in the same chain has at least m points on each halfspace. If we label the points of each chain from 0 to $m-1$ (starting from the convex hull) and consider $p_{i_1}^1 \in C_1, \dots, p_{i_d}^d \in C_d$, they define a $(i_1 + \dots + i_d)$ -facet. Therefore, the number of $(\leq k)$ -facets defined by one point on each of these chains equals the cardinality of the set

$$\{(i_1, \dots, i_d) \in \mathbb{Z}^d : i_1 + \dots + i_d \leq k, 0 \leq i_1, \dots, i_d \leq k\},$$

which is exactly $\binom{k+d}{d}$. Since these are the facets defined by points in d out of the $d+1$ chains, the total number of $(\leq k)$ -facets of the set is exactly $(d+1)\binom{k+d}{d}$. \square

5 Conclusions and open problems

For $S \subset \mathbb{R}^2$ we have shown that, for a fixed $k \leq \lfloor \frac{n}{3} \rfloor - 1$, if $E_k(S)$ is optimal, i.e. $E_k(S) = 3\binom{k+2}{2}$, then $E_j(S)$ is also optimal in the whole range $0 \leq j \leq k$, which in turn implies that $e_j(S) = 3(j+1)$ for $0 \leq j \leq k$. Moreover, then the outermost $\lceil \frac{k}{2} \rceil$ layers of S are triangles and these layers consist entirely of j -edges of special types. In addition, we have been able to give a simple construction showing that the lower bound in Equation 3 is tight for $0 \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$.

All these results reveal significant deeper insight into the structure of sets minimizing the number of k -edges, the final goal being to find tight bounds for every k .

Moreover, for an n -point set $S \subset \mathbb{R}^d$ we have proven the lower bound $(d+1)\binom{k+d}{d}$ for the number of $(\leq k)$ -facets in the range $0 \leq k < \lfloor n/(d+1) \rfloor$, which is the first result of this kind in \mathbb{R}^d .

The restriction $k < \lfloor n/(d+1) \rfloor$ stems from the underlying technique, namely using the centerpoint of a set, and can probably be removed. An alternative proof of Theorem 11, using a simplicial half-net instead of a centerpoint, would be sufficient to extend the bound to the whole range of k . Therefore, it is a challenging task to extend Lemma 3 to dimension d , as the following conjecture states:

Conjecture 12. *Every point set $S \subset \mathbb{R}^d$ has a simplicial half-net.*

References

- [1] B. M. Ábrego and S. Fernández-Merchant. A lower bound for the rectilinear crossing number. *Graphs and Combinatorics*, 21:3 (2005), 293–300.

- [2] O. Aichholzer, J. García, D. Orden and P. A. Ramos. New lower bounds for the number of $(\leq k)$ -edges and the rectilinear crossing number of K_n . *Discrete and Computational Geometry* 38:1 (2007), 1–14.
- [3] A. Andrzejak, B. Aronov, S. Har-Peled, R. Seidel and E. Welzl. Results on k -sets and j -facets via continuous motion. In *Proceedings of the 14th ACM Symposium on Computational Geometry (SoCG)*, Minneapolis, Minnesota, United States, (1998), 192–199.
- [4] J. Balogh and G. Salazar. Improved bounds for the number of $(\leq k)$ -sets, convex quadrilaterals, and the rectilinear crossing number of K_n . In *Proceedings of the 12th International Symposium on Graph Drawing*. Lecture Notes in Computer Science 3383 (2005), 25–35.
- [5] P. Brass, W. Moser and J. Pach. Research problems in discrete geometry. Springer Verlag, 2005.
- [6] H. Edelsbrunner. Algorithms in Combinatorial Geometry, Springer Verlag, 1987.
- [7] H. Edelsbrunner, N. Hasan, R. Seidel and X. J. Shen. Circles through two points that always enclose many points. *Geometriae Dedicata*, 32 (1989), 1–12.
- [8] D. Haussler, and E. Welzl. Epsilon-nets and simplex range queries. *Discrete and Computational Geometry* 2 (2007), 127–151.
- [9] L. Lovász, K. Vesztegombi, U. Wagner, and E. Welzl. Convex Quadrilaterals and k -Sets. In *Towards a Theory of Geometric Graphs*, J. Pach (Ed.) Contemporary Mathematics 342 (2004), 139–148.
- [10] I. Vrt'o, Crossing numbers of graphs: A bibliography.
<http://www.ifi.savba.sk/~imrich>

New results on lower bounds for the number of $(\leq k)$ -facets ^{*}

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Abstract

In this paper we present three different results dealing with the number of $(\leq k)$ -facets of a set of points:

1. We give structural properties of sets in the plane that achieve the optimal lower bound $3\binom{k+2}{2}$ of $(\leq k)$ -edges for a fixed $0 \leq k \leq \lfloor n/3 \rfloor - 1$;
2. We give a simple construction showing that the lower bound $3\binom{k+2}{2} + 3\binom{k - \lfloor \frac{n}{3} \rfloor + 2}{2}$ for the number of $(\leq k)$ -edges of a planar point set appeared in [Aichholzer et al. New lower bounds for the number of $(\leq k)$ -edges and the rectilinear crossing number of K_n . *Disc. Comput. Geom.* 38:1 (2007), 1–14] is optimal in the range $\lfloor n/3 \rfloor \leq k \leq \lfloor 5n/12 \rfloor - 1$;
3. We show that for $k < \lfloor n/(d+1) \rfloor$ the number of $(\leq k)$ -facets of a set of n points in general position in \mathbb{R}^d is at least $(d+1)\binom{k+d}{d}$, and that this bound is tight in the given range of k .

1 Introduction

In this paper we deal with the problem of giving lower bounds to the number of $(\leq k)$ -facets of a set of points S : An oriented simplex with vertices at points of S is said to be a k -facet of S if it has exactly k points in the positive side of its affine hull. Similarly, the

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simplex is said to be an $(\leq k)$ -*facet* if it has at most k points in the positive side of its affine hull. If $S \subset \mathbb{R}^2$, a k -facet of S is usually named a k -edge.

The number of k -facets of S is denoted by $e_k(S)$, and $E_k(S) = \sum_{j=0}^k e_j(S)$ is the number of $(\leq k)$ -facets (the set S will be omitted when it is clear from the context). Giving bounds on these quantities, and on the number of the companion concept of k -sets, is one of the central problems in Discrete and Computational Geometry, and has a long history that we will not try to summarize here. Chapter 8.3 in [5] is a complete and up to date survey of results and open problems in the area.

Regarding lower bounds for $E_k(S)$, which is the main topic of this paper, the problem was first studied by Edelsbrunner et al. [7] due to its connections with the complexity of higher order Voronoi diagrams. In that paper it was stated that, in \mathbb{R}^2 ,

$$E_k(S) \geq 3 \binom{k+2}{2} \quad (1)$$

and it was given an example showing tightness for $0 \leq k \leq \lfloor n/3 \rfloor - 1$. The proof used circular sequences but, unfortunately, contained an unpluggable gap, as pointed out by Lovász et al. [9]. A correct proof, also using circular sequences, was independently found by Ábrego and Fernández-Merchant [1] and Lovász et al. [9]. In both papers a strong connection was discovered between the number of $(\leq k)$ -edges and the number of convex quadrilaterals in a point set S . Specifically, if $\square(S)$ denotes the number of convex quadrilaterals in S , in [9] it was shown that

$$\square(S) = \sum_{k < \frac{n-2}{2}} (n - 2k - 3) E_k(S) - \frac{3}{4} \binom{n}{3} + c_n, \quad (2)$$

where

$$c_n = \begin{cases} \frac{1}{4} E_{\frac{n-3}{2}}(S), & \text{if } n \text{ is odd,} \\ 0, & \text{if } n \text{ is even.} \end{cases}$$

Giving lower bounds for $\square(S)$ is in turn equivalent to determining the rectilinear crossing number of the complete graph: if we draw K_n on top of a set of points S , then the number of intersections in the drawing is exactly the number of convex quadrilaterals in S . The interested reader can go through the extensive online bibliography by Vrt'o [10] where the focus is on the problem of crossing numbers of graphs.

The lower bound in Equation 1 was slightly improved for $k \geq \lfloor \frac{n}{3} \rfloor$ by Balogh and Salazar [4], again using circular sequences. Using different techniques, and based on the observation that it suffices to proof the bound for sets with triangular convex hull, we have recently shown [2] that, in \mathbb{R}^2 ,

$$E_k(S) \geq 3 \binom{k+2}{2} + \sum_{j=\lfloor \frac{n}{3} \rfloor}^k (3j - n + 3). \quad (3)$$

If n is divisible by 3, this expression can be written as

$$E_k(S) \geq 3 \binom{k+2}{2} + \binom{k - \frac{n}{3} + 2}{2}.$$

In this paper we deal with three different problems related to lower bounds for E_k : In Section 2, we study structural properties of those sets in \mathbb{R}^2 that achieve the lower

bound in Equation 1 for a fixed $0 \leq k \leq \lfloor n/3 \rfloor - 1$. The main result of this section is that if $E_k(S)$ is minimum for a given k , then $E_j(S)$ is also minimum for every $0 \leq j < k$. In Section 3 we give a construction which shows tightness of the lower bound in Equation 3 in the range $\lfloor n/3 \rfloor \leq k \leq \lfloor 5n/12 \rfloor - 1$. Finally, in Section 4 we study the d -dimensional version of the problem and show that, for a set of n points in general position in \mathbb{R}^d ,

$$E_k(S) \geq (d+1) \binom{k+d}{d}, \text{ for } 0 \leq k < \lfloor \frac{n}{d+1} \rfloor, \quad (4)$$

and that this bound is tight in that range. To the best of our knowledge, this is the first result of this kind in \mathbb{R}^d .

2 Optimal sets for $(\leq k)$ -edge vectors

Given $S \subset \mathbb{R}^2$, let us denote by $\mathcal{E}_k(S)$ the set of all $(\leq k)$ -edges of S , hence $E_k(S)$ is the cardinality of $\mathcal{E}_k(S)$. Throughout this section we consider $k \leq \lfloor \frac{n}{3} \rfloor - 1$. Recall that for a fixed such k , $E_k(S)$ is optimal if $E_k(S) = 3 \binom{k+2}{2}$. Recall also that, by definition, a j -edge has exactly j points of S in the positive side of its affine hull, which in this case is the open half plane to the right of its supporting line.

We start by giving a new, simple, and self-contained proof of the bound in Equation 1, using a new technique which will be useful in the rest of the section. Although in this section they will be used in \mathbb{R}^2 , the following notions are presented in \mathbb{R}^d for the sake of generality and in view of Chapter 4.

Definition 1 ([8]). Let S be a set of n points and \mathcal{H} a family of sets in \mathbb{R}^d . A subset $N \subset S$ is called an ϵ -net of S (with respect to \mathcal{H}) if for every $H \in \mathcal{H}$ such that $|H \cap S| > \epsilon n$ we have that $H \cap N \neq \emptyset$.

Definition 2. A *simplicial ϵ -net* of $S \subset \mathbb{R}^d$ is a set of $d+1$ vertices of the convex hull of S that are an ϵ -net of S with respect to closed half-spaces. A simplicial $\frac{1}{2}$ -net will be called a *simplicial half-net*.

Lemma 3. *Every set $S \subset \mathbb{R}^2$ of n points has a simplicial half-net.*

Proof. Let T be a triangle spanned by three vertices of the convex hull of S . An edge e of T is called *good* if the closed half plane of its supporting line which contains the third vertex of T , contains at least $\frac{n}{2}$ points from S . T is called *good* if it consists of three good edges. Clearly, the vertices of a good triangle are a simplicial half-net of S .

Let T be an arbitrary triangle spanned by vertices of the convex hull of S and assume that T is not good. Then observe that only one edge e of T is not good and let v be the vertex of T not incident to e . Choose a point v' of the convex hull of S opposite to v with respect to e . Then e and v' induce a triangle T' in which e is a good edge. If T' is a good triangle we are done. Otherwise we iterate this process. As the cardinalities of the subsets of vertices of S considered are strictly decreasing (the subsets being restricted by the half plane induced by e), the process terminates with a good triangle. \square

Theorem 4. *For every set S of n points and $0 \leq k < \lfloor \frac{n-2}{2} \rfloor$ we have $E_k(S) \geq 3 \binom{k+2}{2}$.*

Proof. The proof goes by induction on n . From Lemma 3, we can guarantee the existence of $T = \{a, b, c\} \subset S$, an $\frac{1}{2}$ -net made up with vertices of the convex hull.

Let $S' = S \setminus T$ and consider an edge $e \in \mathcal{E}_{k-2}(S')$. We observe that T cannot be to the right of e : there are at least $\frac{n}{2}$ points on the closed half-plane to the left of e and that would contradict the definition of $\frac{1}{2}$ -net. Therefore, $e \in \mathcal{E}_k(S)$.

If we denote by $\mathcal{ET}_k(S)$ the set of $(\leq k)$ -edges of S adjacent to points in T , we have that

$$\mathcal{E}_{k-2}(S') \cup \mathcal{ET}_k(S) \subset \mathcal{E}_k(S). \quad (5)$$

There are $2(k+1)$ $(\leq k)$ -edges incident to each of the convex hull vertices a, b, c (which can be obtained rotating a ray based on that vertex). We observe that at most three edges of $\mathcal{ET}_k(S)$ might be incident to two points of T (those of the triangle T) and that the union in Equation 5 is disjoint. Therefore, using the induction hypothesis we have

$$E_k(S) \geq E_{k-2}(S') + 3 + 6k \geq 3 \binom{k}{2} + 3 + 6k = 3 \binom{k+2}{2}. \quad (6)$$

□

Corollary 5. *Let S be a set of n points, $T = \{a, b, c\}$ a simplicial half-net of S and $S' = S \setminus T$. If $E_k(S) = 3 \binom{k+2}{2}$ then:*

(a) $E_{k-2}(S') = 3 \binom{k}{2}$.

(b) A k -edge of S is either a $(k-2)$ -edge of S' or is adjacent to a point in T .

Proof. If $E_k(S) = 3 \binom{k+2}{2}$, both inequalities in Equation 6 are tight. Therefore $E_{k-2}(S') = 3 \binom{k}{2}$ and Equation 5 becomes $\mathcal{E}_{k-2}(S') \cup \mathcal{ET}_k(S) = \mathcal{E}_k(S)$ (disjoint union) which trivially implies part (b). □

Theorem 6. *If $E_k(S) = 3 \binom{k+2}{2}$, then S has a triangular convex hull.*

Proof. We prove the statement by induction over k . For $k = 0$ nothing has to be proven, so let $k = 1$, assume that $E_1 = 9$ and let $h = |CH(S)|$. We have h 0-edges and at least h 1-edges (two per convex hull vertex, but each edge might be counted twice). Thus $E_1 = 9 \geq 2h$ and therefore $h \leq 4$. Assume now $h = 4$. Then at most two 1-edges can be counted twice, namely the two diagonals of the convex hull. Thus we have $4 + 8 - 2 = 10$ (≤ 1) -edges and we conclude that if $E_1 = 9$, then S has a triangular convex hull.

For the general case consider $k \geq 2$, let $T = \{a, b, c\}$ be the simplicial half-net guaranteed by Lemma 3 and let $S' = S \setminus T$. From Corollary 5, part (a), we know that $E_{k-2}(S') = 3 \binom{k}{2}$ and, by induction, we may assume that S' has a triangular convex hull. Moreover, from part (b), no $(k-1)$ -edge of S' can be an $(\leq k)$ -edge of S and, therefore, any $(k-1)$ -edge of S' must have two vertices of T on its positive side. Consider the six $(k-1)$ -edges of S' incident to the three convex hull vertices of S' : See Figure 1, where the supporting lines of these $(k-1)$ -edges are drawn as dashed lines and S' is depicted as the central triangle. Each cell outside S' in the arrangement of the supporting lines contains a number counting the $(k-1)$ -edges considered which have that cell on their positive side. A simple counting argument shows that the only way of placing the three vertices a, b, c of T such that each $(k-1)$ -edge of S' drawn has three of them on its positive side is to place one in each cell labeled with a 4. We conclude that no vertex of S' can be on the convex hull of S and the theorem follows. □

Corollary 7. *If $E_k(S) = 3 \binom{k+2}{2}$, then the outermost $\lceil \frac{k}{2} \rceil$ layers of S are triangles.*

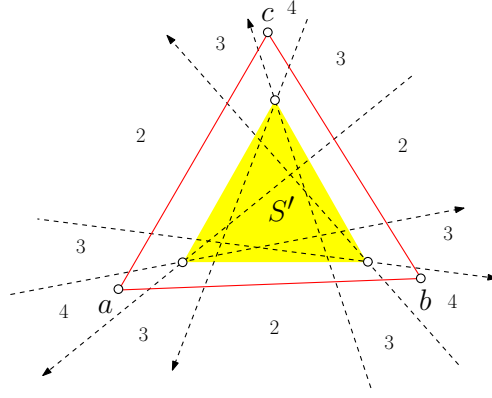


Figure 1: Each $(k-1)$ -edge of S' incident to a convex hull vertex of S' (supporting lines are shown as dashed lines) has two vertices of T on its positive side.

Proof. From the optimality of $E_k(S)$ and using the same argument as in the proof of Theorem 6, it follows that we can iteratively remove the outermost $\lceil \frac{k}{2} \rceil$ layers to obtain optimal subsets, which, by Theorem 6, have triangular convex hulls. \square

Theorem 8. *If $E_k(S) = 3\binom{k+2}{2}$, then $E_j(S) = 3\binom{j+2}{2}$ for every $0 \leq j \leq k$.*

Proof. We prove the theorem by induction on k . For $k = 0, 1$ the theorem is equivalent to Theorem 6, so let $k \geq 2$. It is sufficient to show that optimality of $E_k(S)$ implies optimality of $E_{k-1}(S)$, as the theorem follows by induction.

Let T be the vertices of $CH(S)$ (which is a triangle as guaranteed by Theorem 6) and let $S' = S \setminus T$. As in Theorem 4 we have

$$\mathcal{E}_{k-3}(S') \cup \mathcal{ET}_{k-1}(S) \subset \mathcal{E}_{k-1}(S).$$

Observe that $E_{k-2}(S')$ is optimal, as guaranteed by Corollary 5 and this implies optimality of $E_{k-3}(S')$ by induction. $|\mathcal{ET}_{k-1}(S)|$ is also optimal because the convex hull of S is the triangle T . Therefore, to prove optimality of $E_{k-1}(S)$ it only remains to show that no $(k-2)$ -edge of S' can be a $(k-1)$ -edge of S .

So let e be a $(k-2)$ -edge of S' and let p and q be the vertices of the convex hull of S' incident to e or on its positive side. The existence of p and q is guaranteed by Corollary 5, part (b). Without loss of generality, assume that the edge pq is horizontal with the remaining vertices of S' above it, see Figure 2 for the rest of the proof. Let ℓ_1 be the $(k-1)$ -edge of S' incident to p which has q on its positive side and ℓ_2 the $(k-1)$ -edge incident to q and having p on its positive side. The *boundary chain* is the lower envelope of ℓ_1 , pq , and ℓ_2 . We claim that e does not intersect the boundary chain and lies above it. If e is incident to p or q then the claim is obviously true. Otherwise observe that e has to intersect the supporting lines of both considered $(k-1)$ -edges in the interior of S' , as otherwise there would be too many vertices on the positive side of e . But then again e lies above the boundary chain and the claim follows.

From the proof of Theorem 6 we know that two of the vertices of the convex hull of S have to lie below our boundary chain (below the $(k-1)$ -edges, see a and b in Figure 2) and thus on the positive side of e . Therefore e has at least k vertices of S on its positive side and does not belong to $\mathcal{E}_{k-1}(S)$. We conclude that $E_{k-1}(S)$ is optimal and the theorem follows. \square

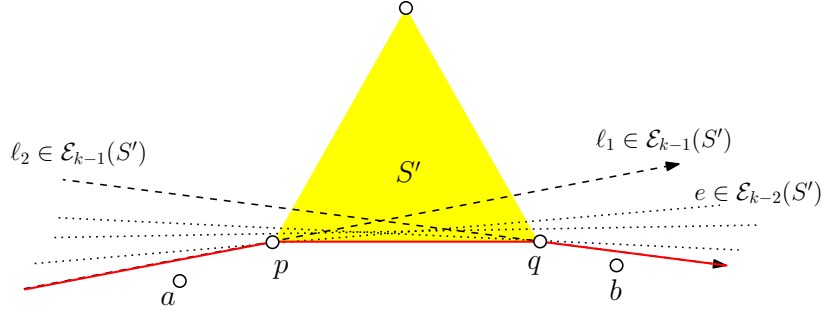


Figure 2: All $(k-2)$ -edges of S' (supporting lines are shown as dotted lines) lie above the (bold) lower envelope.

Corollary 9. *Let $0 \leq k \leq \lfloor \frac{n}{3} \rfloor - 1$. If $E_k(S) = 3 \binom{k+2}{2}$, then $e_j(S) = 3(j+1)$ for $0 \leq j \leq k$.*

3 Tightness of the lower bound for $(\leq k)$ -edges in \mathbb{R}^2

In this section we show a point configuration which proves tightness in the range $0 \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$ of the lower bound for $E_k(S)$ given in [2]. Consider the configuration in Figure 3 (left), which is composed of three rotationally symmetric chains, each one associated to a convex hull vertex, fulfilling the following properties (where left and right are considered with respect to the corresponding convex hull vertex):

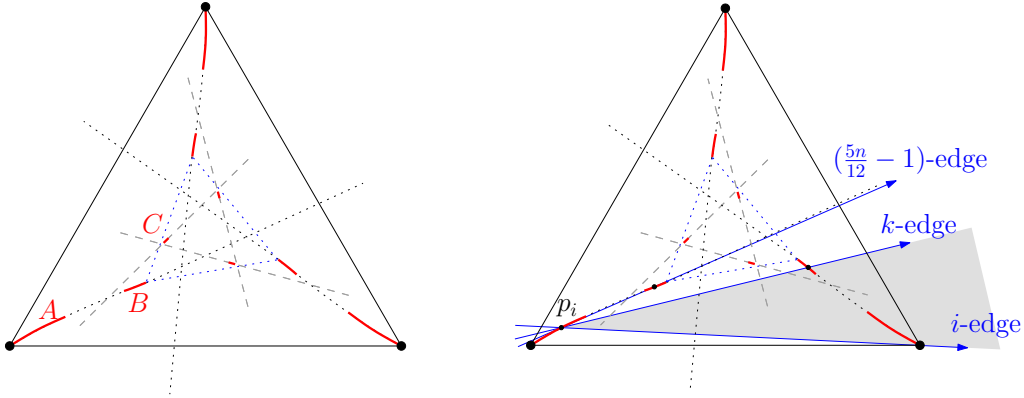


Figure 3: Left: Configuration showing tightness for $0 \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$. Right: For $i \in \{0, \dots, \frac{2n}{12} - 1\}$, exactly one j -edge appears for each $j \in \{i, \dots, k\}$.

- The first part of the chain is slightly convex to the right and contains $\frac{3n}{12}$ points, with a hole between the first $\frac{2n}{12}$ points (which we call subchain A) and the remaining $\frac{n}{12}$ points (called subchain B).
- Each chain is completed with a subchain C , composed of another $\frac{n}{12}$ points slightly convex to the left.
- All the lines spanned by two points in $A \cup B$, oriented from the outermost to the innermost endpoint, leave to the right the next chain in counterclockwise order, and to the left both the points in C and those in the remaining chain.

- All the lines spanned by two points in C separate subchains A and B . Furthermore, when oriented from the outermost to the innermost endpoint, they leave to the right the two remaining subchains of type C .
- The triangle defined by the innermost points of chains of type B contains all the chains of type C .

Theorem 10. For the point configuration S defined above and $\lfloor \frac{n}{3} \rfloor \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$,

$$E_k(S) = 3 \binom{k+2}{2} + 3 \binom{k - \frac{n}{3} + 2}{2},$$

which matches the lower bound stated in [2] for n divisible by 3.

Proof. Because of the rotational symmetry, we can focus on one of the three chains $A \cup B \cup C$ and let p_i be the $(i+1)$ -th point on that chain. We will count oriented j -edges of type $\overrightarrow{p_i q}$ (i.e. with p_i on the tail) for $j \leq k$. In order to do so we rotate counterclockwise a ray based on p_i , starting from the one passing through the convex hull vertex of the next chain in counterclockwise order. Three cases arise, depending on the index i of p_i , which correspond to p_i lying on one of the three subchains:

- (A) For $i \in \{0, \dots, \frac{2n}{12} - 1\}$, exactly one j -edge appears for each j in the range $j \in \{i, \dots, k\}$, while all the remaining j -edges $\overrightarrow{p_i q}$ in the rotation have $j > k$ since at some point we find a $(\frac{5n}{12} - 1)$ -edge and after that all the j -edges found have $j \geq \frac{5n}{12} > k$. See Figure 3 (right).
- (B) For $i \in \{\frac{2n}{12}, \dots, \frac{3n}{12} - 1\}$, exactly one j -edge appears for each j in the ranges $j \in \{i, \dots, k\}$ and $j \in \{\frac{7n}{12} - i - 1, \dots, k\}$, while all the remaining j -edges $\overrightarrow{p_i q}$ in the rotation have $j > k$. See Figure 4.
- (C) For $i \in \{\frac{3n}{12}, \dots, \frac{4n}{12} - 1\}$, exactly one j -edge appears for each j in the ranges $j \in \{i, \dots, k\}$ and $j \in \{\frac{8n}{12} - i - 1, \dots, k\}$, while all the remaining j -edges $\overrightarrow{p_i q}$ in the rotation have $j > k$. See Figure 5.

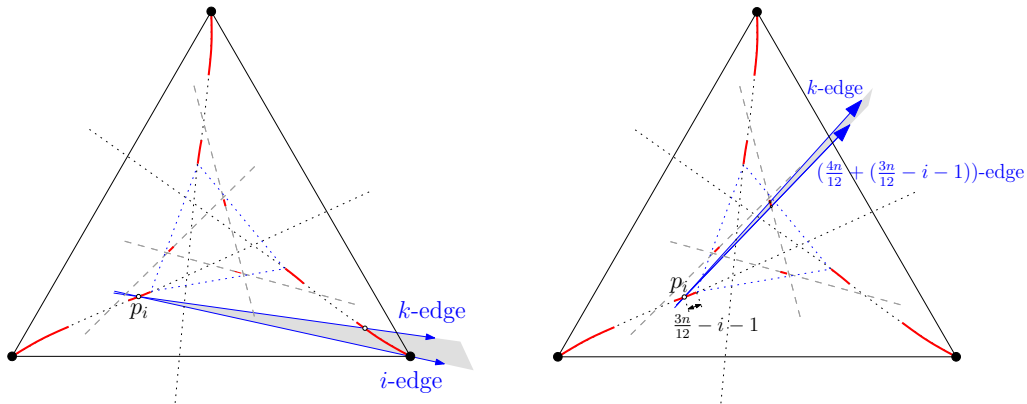


Figure 4: Left: For $i \in \{\frac{2n}{12}, \dots, \frac{3n}{12} - 1\}$, exactly one j -edge appears for each $j \in \{i, \dots, k\}$. Right: For $i \in \{\frac{2n}{12}, \dots, \frac{3n}{12} - 1\}$, exactly one j -edge appears for each $j \in \{\frac{7n}{12} - i - 1, \dots, k\}$.

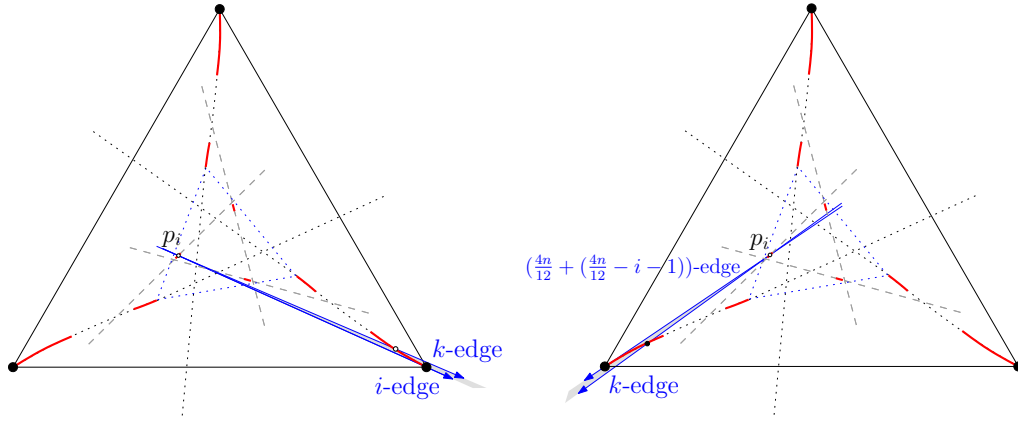


Figure 5: Left: For $i \in \{\frac{3n}{12}, \dots, \frac{4n}{12} - 1\}$, exactly one j -edge appears for each $j \in \{i, \dots, k\}$. Right: For $i \in \{\frac{3n}{12}, \dots, \frac{4n}{12} - 1\}$, exactly one j -edge appears for each $j \in \{\frac{8n}{12} - i - 1, \dots, k\}$.

Let us point out that, depending on the values of k and i , some of the above ranges could actually be empty. Now we are ready to count the total number of $(\leq k)$ -edges incident to points p_i on the first chain, which is:

$$\sum_{i=0}^{\frac{2n}{12}-1} (k-i+1) + \sum_{i=\frac{2n}{12}}^{\frac{3n}{12}-1} (k-i+1) + \sum_{i=\frac{3n}{12}}^{\frac{4n}{12}-1} (k-i+1) + \sum_{i=\frac{2n}{12}}^{\frac{3n}{12}-1} (k - \frac{7n}{12} + i + 2) + \sum_{i=\frac{3n}{12}}^{\frac{4n}{12}-1} (k - \frac{8n}{12} + i + 2),$$

where the first three summands come from the first ranges of the three cases above, while the two remaining summands come from the second ranges in cases (B) and (C). Merging the first three summands and rewriting the two latter ones, the above sum equals

$$\sum_{i=0}^{\frac{4n}{12}-1} (k-i+1) + \sum_{i=\frac{5n}{12}-1}^{\frac{4n}{12}} (k-i+1) + \sum_{i=\frac{5n}{12}-1}^{\frac{4n}{12}} (k-i+1) = \sum_{j=1}^{k+1} j + \sum_{j=1}^{k-\frac{4n}{12}+1} j = \binom{k+2}{2} + \binom{k-\frac{n}{3}+2}{2},$$

where the first equality comes from neglecting the negative summands, due to the above mentioned empty ranges, and merging the first two sums. This result has to be multiplied by the three chains of the configuration, so we get

$$E_k(S) = 3 \binom{k+2}{2} + 3 \binom{k-\frac{n}{3}+2}{2}.$$

□

4 A lower bound for $(\leq k)$ -facets in \mathbb{R}^d

Throughout this section, $S \subset \mathbb{R}^d$ will be a set of n points in general position.

We remind that $e_k(S)$ and $E_k(S)$ denote, respectively, the number of k -facets and the number of $(\leq k)$ -facets of S . The main result of this section is a lower bound for the number of $(\leq k)$ -facets of a set of n points in general position in \mathbb{R}^d in the range $0 \leq k < \lfloor \frac{n}{d+1} \rfloor$.

The proof follows the approach in Theorem 4, using the fact that every set of points has a centerpoint: a point $c \in \mathbb{R}^d$ is a *centerpoint* of S if no open halfspace that avoids c contains more than $\lceil \frac{dn}{d+1} \rceil$ points of S (see [6]).

Theorem 11. *Let S be a set of $n \geq d+1$ points in \mathbb{R}^d in general position. Then*

$$E_k(S) \geq (d+1) \binom{k+d}{d} \quad \text{if } 0 \leq k < \lfloor \frac{n}{d+1} \rfloor.$$

Furthermore, the bound on $E_k(S)$ is tight in the given range of k .

Proof. The proof uses induction on n and d . The base case for $n = d+1$ is obvious and for $d = 2$ is just Equation 1.

Let $k < \lfloor \frac{n}{d+1} \rfloor$ and let c be a centerpoint of S . Let us consider a simplex T with vertices in the convex hull of S and containing c and let $S' = S \setminus T$. From the definition of centerpoint, it follows that no open halfspace that avoids T contains more than $\lceil \frac{dn}{d+1} \rceil - 1$ points or, equivalently, every closed halfspace containing T has at least $\lfloor \frac{n}{d+1} \rfloor + 1$ points.

We denote by $\mathcal{E}_k^j(S)$ the set of $(\leq k)$ -facets of S adjacent to exactly j vertices of T , and $E_k^j(S)$ will be the cardinality of $\mathcal{E}_k^j(S)$.

For $j = 0$, we observe that $\mathcal{E}_{k-d}^0(S') \subset \mathcal{E}_k^0(S)$, because a closed halfspace containing at most k points cannot contain all the vertices of T . Because $k-d \leq \lfloor \frac{n-(d+1)}{d+1} \rfloor - 1$, we can apply induction on n and get

$$E_k^0(S) \geq E_{k-d}^0(S') \geq (d+1) \binom{(k-d)+d}{d} = (d+1) \binom{k}{d}.$$

For $1 \leq j \leq d$, let T_j be a subset of j vertices of T and let S_π be the projection from T_j of $S \setminus T$ onto the $(d-j)$ -dimensional subspace π defined by the points in $T \setminus T_j$: a point $p \in S \setminus T$ is mapped to the intersection between the j -flat defined by p and T_j and the $(d-j)$ -flat defined by points in $T \setminus T_j$. Using the general position assumption, it is easy to see that the intersection has dimension zero. If the intersection were empty, we could slightly perturb p without changing the number of $(\leq k)$ -facets of S .

Now, if $\sigma \subset S_\pi$ is an $(\leq (k-d+j))$ -facet of S_π , then $\sigma \cup T_j$ is an $(\leq k)$ -facet of S (as before, a halfspace containing at most k points of S cannot contain all the vertices of T). Because

$$k-d+j \leq \left\lfloor \frac{n}{d+1} \right\rfloor - 1 \leq \left\lfloor \frac{n-j}{d-j+1} \right\rfloor - 1$$

we can apply induction in d and n , obtaining that there are at least

$$(d-j+1) \binom{k-d+j+(d-j)}{d-j} = (d-j+1) \binom{k}{d-j}$$

$(\leq k)$ -facets of S adjacent to T_j . Summing on all the subsets of j points of T , we get

$$E_k^j(S) \geq \binom{d+1}{j} (d-j+1) \binom{k}{d-j},$$

and, finally,

$$E_k(S) \geq \sum_{j=0}^d \binom{d+1}{j} (d-j+1) \binom{k}{d-j} = (d+1) \binom{k+d}{d}.$$

As for tightness, the example showing that the bound $3\binom{k+2}{2}$ is tight for $0 \leq k \leq \lfloor \frac{n}{3} \rfloor - 1$ in the planar case [7] can be extended to \mathbb{R}^d : Consider $d+1$ rays in \mathbb{R}^d emanating from the origin and with the property that any hyperplane containing one of them leaves on each open halfspace at least one of the remaining rays. For instance, we could take the rays defined by the origin and the vertices of a regular simplex inscribed in the unit d -sphere.

Let $n = (d+1)m$ and put chains C_1, \dots, C_{d+1} with m points on each ray, slightly perturbed to achieve general position. For $j < m$, every j -facet of S is defined by d points on different chains, because a facet defined by two points in the same chain has at least m points on each halfspace. If we label the points of each chain from 0 to $m-1$ (starting from the convex hull) and consider $p_{i_1}^1 \in C_1, \dots, p_{i_d}^d \in C_d$, they define a $(i_1 + \dots + i_d)$ -facet. Therefore, the number of $(\leq k)$ -facets defined by one point on each of these chains equals the cardinality of the set

$$\{(i_1, \dots, i_d) \in \mathbb{Z}^d : i_1 + \dots + i_d \leq k, 0 \leq i_1, \dots, i_d \leq k\},$$

which is exactly $\binom{k+d}{d}$. Since these are the facets defined by points in d out of the $d+1$ chains, the total number of $(\leq k)$ -facets of the set is exactly $(d+1)\binom{k+d}{d}$. \square

5 Conclusions and open problems

For $S \subset \mathbb{R}^2$ we have shown that, for a fixed $k \leq \lfloor \frac{n}{3} \rfloor - 1$, if $E_k(S)$ is optimal, i.e. $E_k(S) = 3\binom{k+2}{2}$, then $E_j(S)$ is also optimal in the whole range $0 \leq j \leq k$, which in turn implies that $e_j(S) = 3(j+1)$ for $0 \leq j \leq k$. Moreover, then the outermost $\lceil \frac{k}{2} \rceil$ layers of S are triangles and these layers consist entirely of j -edges of special types. In addition, we have been able to give a simple construction showing that the lower bound in Equation 3 is tight for $0 \leq k \leq \lfloor \frac{5n}{12} \rfloor - 1$.

All these results reveal significant deeper insight into the structure of sets minimizing the number of k -edges, the final goal being to find tight bounds for every k .

Moreover, for an n -point set $S \subset \mathbb{R}^d$ we have proven the lower bound $(d+1)\binom{k+d}{d}$ for the number of $(\leq k)$ -facets in the range $0 \leq k < \lfloor n/(d+1) \rfloor$, which is the first result of this kind in \mathbb{R}^d .

The restriction $k < \lfloor n/(d+1) \rfloor$ stems from the underlying technique, namely using the centerpoint of a set, and can probably be removed. An alternative proof of Theorem 11, using a simplicial half-net instead of a centerpoint, would be sufficient to extend the bound to the whole range of k . Therefore, it is a challenging task to extend Lemma 3 to dimension d , as the following conjecture states:

Conjecture 12. *Every point set $S \subset \mathbb{R}^d$ has a simplicial half-net.*

References

- [1] B. M. Ábrego and S. Fernández-Merchant. A lower bound for the rectilinear crossing number. *Graphs and Combinatorics*, 21:3 (2005), 293–300.

- [2] O. Aichholzer, J. García, D. Orden and P. A. Ramos. New lower bounds for the number of $(\leq k)$ -edges and the rectilinear crossing number of K_n . *Discrete and Computational Geometry* 38:1 (2007), 1–14.
- [3] A. Andrzejak, B. Aronov, S. Har-Peled, R. Seidel and E. Welzl. Results on k -sets and j -facets via continuous motion. In *Proceedings of the 14th ACM Symposium on Computational Geometry (SoCG)*, Minneapolis, Minnesota, United States, (1998), 192–199.
- [4] J. Balogh and G. Salazar. Improved bounds for the number of $(\leq k)$ -sets, convex quadrilaterals, and the rectilinear crossing number of K_n . In *Proceedings of the 12th International Symposium on Graph Drawing*. Lecture Notes in Computer Science 3383 (2005), 25–35.
- [5] P. Brass, W. Moser and J. Pach. Research problems in discrete geometry. Springer Verlag, 2005.
- [6] H. Edelsbrunner. Algorithms in Combinatorial Geometry, Springer Verlag, 1987.
- [7] H. Edelsbrunner, N. Hasan, R. Seidel and X. J. Shen. Circles through two points that always enclose many points. *Geometriae Dedicata*, 32 (1989), 1–12.
- [8] D. Haussler, and E. Welzl. Epsilon-nets and simplex range queries. *Discrete and Computational Geometry* 2 (2007), 127–151.
- [9] L. Lovász, K. Vesztegombi, U. Wagner, and E. Welzl. Convex Quadrilaterals and k -Sets. In *Towards a Theory of Geometric Graphs*, J. Pach (Ed.) Contemporary Mathematics 342 (2004), 139–148.
- [10] I. Vrt’o, Crossing numbers of graphs: A bibliography.
<http://www.ifi.savba.sk/~imrich>